Complex Analysis

Study of the set of complex numbers & functions of complex variables.

$\mathbb{N}$ = the set of natural numbers
$= \{1, 2, 3, \ldots\}$

$\mathbb{Z}$ = the set of all integers.

$\mathbb{Q} = \{ \frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0 \}$

$\mathbb{R}$ = the set of real numbers

$\mathbb{C} = \{ x + iy : x, y \in \mathbb{R} \}$ = the set of complex numbers.

i is just a symbol.

$i^2 = -1$.

If $x, y \in \mathbb{R}$, if $z = x + iy$, we say z is <u>imaginary</u> if $x = 0$.

z is <u>real</u> if $y = 0$.

The complex conjugate of z $= \bar{z}$
$= x - iy$

For example, $\overline{3-4i} = 3+4i$

$$z = 3-4i = x+iy$$
$$x = 3$$
$$y = -4.$$

Note that $\overline{\overline{z}} = z$.

We rig the complex numbers to have the same algebraic properties as real numbers:

Addition: $(a+bi)+(c+di)$
$= (a+c)+(b+d)i \quad \forall a,b,c,d \in \mathbb{R}.$

Multiplication: $(a+bi)(c+di)$

$$= ac + bdi^2 + (bc+ad)i$$
$$= (ac-bd) + (bc+ad)i.$$
$$\forall a,b,c,d \in \mathbb{R}.$$

Notation: $\mathrm{Re}(z) = x$ if $z = x+iy$ for
$\mathrm{Im}(z) = y$ $x,y \in \mathbb{R}.$

Properties of Complex #s.

Group under +

Abelian group under +

① If $z, w \in \mathbb{C}$, then $z+w \in \mathbb{C}$. Closure of +

② $\forall z,u,v \in \mathbb{C}$, $(z+u)+v = z+(u+v)$.

Assoc. Prop. of +

③ $\forall z \in \mathbb{C}$, $0+z = z = \cancel{z+0}$.

Identity Prop of +

④ $\forall z \in \mathbb{C}$, $\exists w \in \mathbb{C}$ s.t.

$z+w = w+z = 0$. Inverse Prop of +

$(w = -z)$

⑤ $\forall z,w \in \mathbb{C}$, $z+w = w+z$.

Commutative Prop of +

Multiplication Properties

⑥ If $z, w \in \mathbb{C}$, then $z \cdot w \in \mathbb{C}$.

Closure for $\cdot$

⑦ If $z, w, u \in \mathbb{C}$, then $(zw)u = z(wu)$.

Assoc. Prop of $\cdot$

⑧ If $z \in \mathbb{C}$, then $1 \cdot z = z = z \cdot 1$

Identity Prop of $\cdot$

⑨ If $z \in \mathbb{C}$, $z \neq 0$, $\exists w \in \mathbb{C}$ s.t.

$zw = 1$ Inverse Prop of $\cdot$

⑩ If $z, w \in \mathbb{C}$, then $w \cdot z = z \cdot w$.

Commutative Property of $\cdot$

Additional Properties.

(ii) If $z, w, u \in \mathbb{C}$, then

$$(z+w)u = zu + wu$$

$$u(z+w) = uz + uw.$$

Distr. Properties

Ex. Find $(3+4i)^{-1}$.

$$(3+4i)(a+bi) = 1$$

$$(3a-4b) + (4a+3b)i = 1 = 1+0i$$

Real parts $3a - 4b = 1$ (1)

Im parts $4a + 3b = 0$ (2)

$\to 4a = -3b \Rightarrow a = \frac{-3}{4}b$ (2b)

(1) & (2b) $\to$ $-\frac{9}{4}b - 4b = 1$

$$\frac{-9-16}{4}b = 1 \Rightarrow -\frac{25}{4}b = 1 \Rightarrow b = \frac{-4}{25}$$

$$a = -\frac{3}{4}b = \frac{-3}{4} \cdot \frac{-4}{25} = \frac{+3}{25}$$

Answer: $\boxed{(3+4i)^{-1} = \frac{3}{25} - \frac{4}{25}i} = \frac{3-4i}{25}$

Useful complex math: If $z = x + iy$, with $x, y \in \mathbb{R}$,

$$z\bar{z} = (x+iy)(x-iy)$$
$$= x^2 - i^2y^2 = x^2 + y^2.$$

We define $|z| = \sqrt{x^2 + y^2}$.

Absolute value or norm of z.

$$\therefore \; z\bar{z} = |z|^2, \quad \forall z \in \mathbb{C}.$$

$$(3+4i)^{-1} = \frac{1}{3+4i} = \frac{1}{(3+4i)}\frac{(3-4i)}{(3-4i)}$$

$$= \frac{3-4i}{3^2+4^2} = \boxed{\frac{3-4i}{25}}$$

Why do we call this an absolute value? $|x+iy| = \sqrt{x^2+y^2}$.

It behaves algebraically like the absolute value. $\forall z \in \mathbb{C}, \; w \in \mathbb{C}$

1. $|z| \geq 0$, $|z| = 0$ if and only if $z = 0$.
2. $|z+w| \leq |z| + |w|$ triangle inequality
3. $|zw| = |z| \cdot |w|$

Other algebraic identities for complex numbers.

- $|\bar{z}| = |z|$.

  Proof: If $z = x+iy$ with $x, y \in \mathbb{R}$, then $|\bar{z}| = |x - iy| = \sqrt{x^2 + (-y)^2} = \sqrt{x^2+y^2} = |z|$. $\boxed{}$

- Ⓐ $\overline{z+w} = \bar{z} + \bar{w}$

  Ⓑ $\overline{zw} = \bar{z} \cdot \bar{w}$.

Proof of B: If $z = a+bi$, $w = c+di$ with $a, b, c, d \in \mathbb{R}$, then

$$\overline{zw} = \overline{(a+bi)(c+di)} = \overline{(ac-bd) + (bc+ad)i}$$
$$= (ac-bd) + (-bc-ad)i.$$

On the other hand

$$\bar{z}\bar{w} = (a-bi)(c-di) = (ac-bd) + (-bc-ad)i.$$
$$= \overline{zw}. \quad \boxed{}$$

Proof of $|zw| = |z||w|$: $\forall z, w \in \mathbb{C}$,

$$|zw|^2 \overset{\text{since } |A|^2 = A\bar{A}}{=} (zw)(\overline{zw}) \overset{\text{since } \overline{zw} = \bar{z}\bar{w}}{=} (zw)(\bar{z}\bar{w}) \overset{\text{Commut. of } \cdot \text{, assoc. of } \cdot}{=} z\bar{z}\, w\bar{w}$$
$$= |z|^2 |w|^2.$$

Take $\sqrt{\ }$ of both sides: $|zw| = |z||w|$

(Details: $\sqrt{|zw|^2} = \sqrt{|z|^2|w|^2} = \sqrt{|z|^2}\sqrt{|w|^2}$

real # fact. $\sqrt{AB} = \sqrt{A}\sqrt{B}$ for positive A, B.

$\Rightarrow \quad |zw| = |z|\,|w|.$)

Another example: Proof that $|z+w| \le |z| + |w|$ $\forall z, w \in \mathbb{C}$

Pf. $\forall z, w \in \mathbb{C}$,

$$|z+w|^2 = (z+w)\overline{(z+w)} = (z+w)(\bar{z}+\bar{w})$$

$$= z\bar{z} + \underbrace{w\bar{z} + z\bar{w}} + w\bar{w} \quad (**)$$

Consider $w\bar{z} + z\bar{w} = \bar{z}w + z\bar{w}$

If $A \in \mathbb{C}$, $A + \bar{A} = 2\operatorname{Re}(A)$. $\quad = 2\operatorname{Re}(z\bar{w})$

Next if $B \in \mathbb{C}$, $\quad |B| \ge |\operatorname{Re} B|$

$B = x+iy$, $x, y \in \mathbb{R}$ $\quad \sqrt{x^2+y^2} \ge |x|$

Thus, $w\bar{z} + z\bar{w} = 2\operatorname{Re}(z\bar{w}) \in \mathbb{R}$

$$\Rightarrow |w\bar{z} + z\bar{w}| = 2|\operatorname{Re}(z\bar{w})| \le 2|z\bar{w}|$$

$$= 2|z||\bar{w}| = 2|z|\,|w| \quad (***)$$

$(**) + (***) \Rightarrow$

$$|z+w|^2 = |z|^2 + 2\operatorname{Re}(z\bar{w}) + |w|^2 = \Big|\underbrace{|z|^2}_{\in\mathbb{R}} + \underbrace{2\operatorname{Re}(z\bar{w})}_{\in\mathbb{R}} + \underbrace{|w|^2}_{\in\mathbb{R}}\Big|$$

$$\leq \left||z|^2\right| + 2\left|\mathrm{Re}(z\bar{w})\right| + \left||w|^2\right|$$

Real # triangle inequality.

$$\leq |z|^2 + 2|z||w| + |w|^2$$

$$= \left(|z| + |w|\right)^2.$$

i.e. $|z+w|^2 \leq \left(|z|+|w|\right)^2$

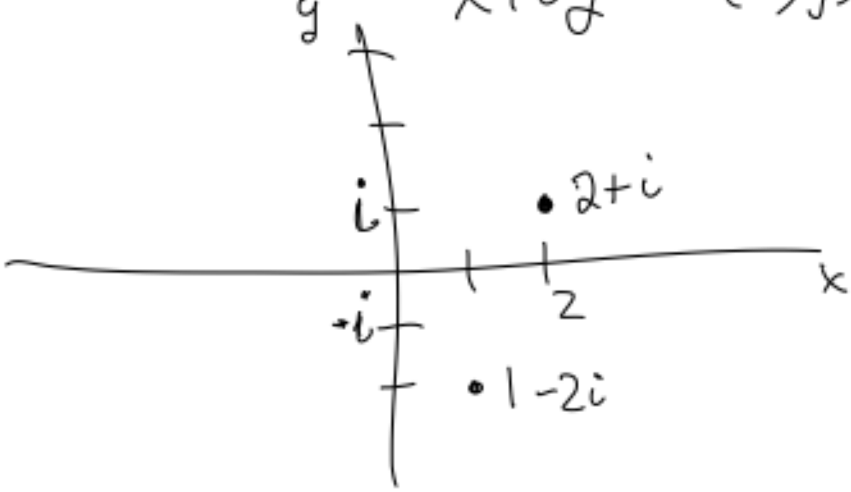
Take $\sqrt{\ }$ of both sides. Since $\sqrt{x}$ is an increasing fcn, it preserves inequalities, so

$x \in [0,\infty)$

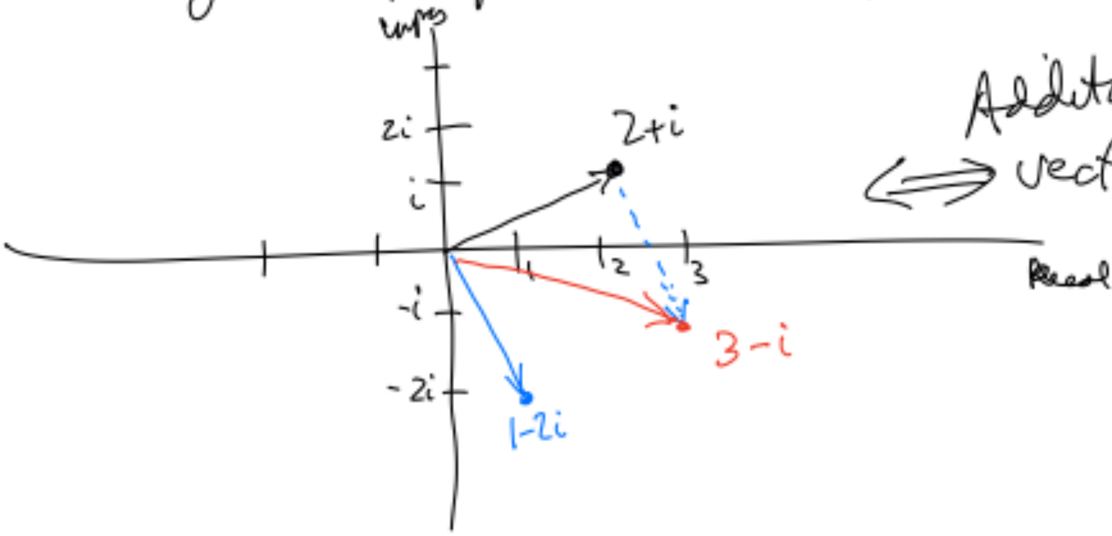
$$\sqrt{|z+w|^2} \leq \sqrt{\left(|z|+|w|\right)^2}$$

$$\Rightarrow |z+w| \leq |z| + |w|. \quad \blacksquare$$

Geometry side — we think of complex numbers as living in $\mathbb{R}^2 = \mathbb{C}$ where $x+iy = (x,y)$.



Algebraic operations corresponding to geometric operations on the plane.



Addition in $\mathbb{C}$ $\Longleftrightarrow$ vector addition.